Exercise 1

Find the spherical coordinates of the Cartesian point $(\sqrt{2}, -\sqrt{6}, -2\sqrt{2})$.

Solution

The relationship between spherical coordinates (ρ, θ, ϕ), ϕ being the polar angle, and Cartesian coordinates is

$$x = \rho \sin\phi \cos\theta \tag{1}$$
$$y = \rho \sin\phi \sin\theta \tag{2}$$
$$z = \rho \cos\phi. \tag{3}$$

To get ρ, square both sides of each equation and add the respective sides together.

$$\begin{aligned} x^2 + y^2 + z^2 &= (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 + (\rho\cos\phi)^2 \\ &= \rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\cos^2\phi \\ &= \rho^2(\sin^2\phi + \cos^2\phi) \\ &= \rho^2 \end{aligned}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Plugging in $x = \sqrt{2}$, $y = -\sqrt{6}$, and $z = -2\sqrt{2}$ results in

$$\rho = 4.$$

To get θ, divide the respective sides of equation (2) by those of equation (1).

$$\frac{y}{x} = \frac{\rho \sin\phi \sin\theta}{\rho \sin\phi \cos\theta} \quad \rightarrow \quad \tan\theta = \frac{y}{x}$$

Plugging in $x = \sqrt{2}$ and $y = -\sqrt{6}$ results in

$$\tan\theta = -\sqrt{3}.$$

$$\theta = \tan^{-1}(-\sqrt{3}) + 2\pi = \frac{5\pi}{3}.$$

Note that 2π is added because $x = \sqrt{2}$ and $y = -\sqrt{6}$ is in the fourth quadrant, and θ needs to be between 0 and 2π. Finally, use equation (3) to determine ϕ.

$$z = \rho\cos\phi \quad \rightarrow \quad \cos\phi = \frac{z}{\rho} = \frac{-2\sqrt{2}}{4} = -\frac{\sqrt{2}}{2} \quad \rightarrow \quad \phi = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

Therefore, the Cartesian point $(\sqrt{2}, -\sqrt{6}, -2\sqrt{2})$ is written in spherical coordinates as

$$\left(4, \frac{5\pi}{3}, \frac{3\pi}{4}\right).$$